PhD in Economics, Management and Statistics

Game Theory Module

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Outline

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Outline

Introduction
- Aim and Objective
- Expected Learning Outcome
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- Lecture Notes and Contact

Introduction to Games

Strategic Form Games

Extensive Form Games

Repeated Games

- **Aim and Objective**:
  - To provide the basics of the theory of games both from a theoretical point of view and from the application of the main game-theoretical tools to the solution of specific cases.
  - To study and solve games in strategic and extensive form in a context of both a complete and an incomplete information. Nash (1953) solution concept will be our main tool. The module will also encompass the study of repreated games.

Introduction

Expected Learning Outcome

- **Expected Learning Outcome**: Students are expected to be able to:
  - Characterize interaction among economic agents within a game-theoretical framework;
  - Define the equilibrium of the game;
  - Understanding the welfare implication of such an equilibrium.

Introduction

Reading

- **Reading**:
  - Those notes are based on *Osborne M.J. and A. Rubinstein (1994). A Course in Game Theory. MIT Press*. The book can be downloaded for free from A. Rubinstein website (http://arielrubinstein.tau.ac.il/). A brief registration form is required.
- **Additional reading**:

1. Kreps, D. (1990) [DK]. Game theory and economic modelling. Clarendon Press, Oxford
2. Gibbons, R. (1992) [RG]. A primer in game theory. Prentice Hall
3. Myerson, R. (1991)[RM]. Game theory. Analysis of conflict. Harvard University Press

Introduction

Reading

- **Additional reading - Continued**:

4. Fudenberg D. and J. Tirole (1991) [FT]. Game theory. MIT Press
5. Varian, H. (1992) [HV]. Microeonomic Analysis. W. W. Norton &Company
6. Mas-Colell, A., Whinston, M.D. and J.R. Green (1995) [MWG]. Microeconomic theory. Oxford University Press

Introduction

Reading

- DK is a clear and non-technical introduction to games. Ideal for beginners.
- RG offers a more formal treatment. Still, suitable book for beginners.
- RM is a technical textbook with advanced material.
- FT is a bible for game theorist. It covers all the aspects of games. Harder sections are marked with "***". Book for advanced users.
- HV and MWG are microeconomic textbooks. Nonetheless, they contain some chapters on games with many useful insights (expecially MWG).

- You can find those notes on my personal website. The address is:

  https://sites.google.com/site/dmaimoneansaldopatti/teaching/PhD

- If you want to talk to me, we can discuss after the lecture and/or we can arrange a meeting via email (dmaimone@unime.it).

Outline

Introduction

Introduction to Games

- Preliminaries
- Type of Games
- Foundamentals of a Game
- Mathematical Notation

Strategic Form Games

Extensive Form Games

Repeated Games

Introduction to Games

Preliminaries

- Game theory is a set of tools that can be used to analyse and understand the interaction of rational agents involved in a decision-making process.
- The above definition implies a couple of important things:
  1. Agents are assumed to be *rational*;
  2. When taking a decision, individuals should consider their expectation of what the counterpart(s) will decide. In other words, individuals are assumed to *think strategically.*
- Whether the above is true can be questionable in a broad numbers of cases.

- It follows that we need to restrict somehow the above definition. Therefore, we can claim that:

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Definition

Game theory is a set of tools that allow us to analyse and understand how rational and strategically-thinking individuals should behave when involved in a decision-making process along with other agents.

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Definition

Game theory is a set of tools that allow us to analyse and understand how rational and strategically-thinking individuals should behave when involved in a decision-making process along with other agents.

- Game theory does not predict how the interaction among individuals *will* evolve, but how it *would* evolve, if, in fact, agents are rational and think strategically.

Introduction to Games

Preliminaries

- Therefore, game theory can be considered a highly abstract representation of real-life situations.
- Its abstracteness offers the necessary flexibility to use a game-theoretical approach to analyse numerous situations in different context: politics, economics, finance, political science and even biology.
- Although it is mathematically grounded, we study game theory in its social dimension.
- In other words, we will not analyse in details the mathematical properties of games but we will go deeply into their ability of describing real life situations.

Introduction to Games

Type of Games

- Depending on the characteristics of a game, we can distinguish:

1. **Cooperative and non-cooperative games**:
   - If we focus on the possible actions of *players*, the game is likely to be a *non-cooperative* one.
   - Instead a game is *cooperative*, if we pay attention to the possible joint actions of a group of players;
2. **Strategic and extensive games**:
   - If players choose **simultaneously** their action once and for all at the beginning of the game, the game is **strategic**;
   - If players do not decide their plan of actions at the beginning of the game,**but whenever a decision has to be taken** given the presence of a sequence of events, the game is said to be **extensive**.

- Further we can distinguish between:

3. **Games with perfect and imperfect information**:

   - If players are **fully informed** about the structure of the game, the set of other players' potential actions and the consequence of those actions, the game is characterized by **perfect information**.

   - Instead, if some of the above elements **are not (or partially) known** by one or more players, the game is characterized by **imperfect information**.

Introduction to Games

Foundamentals of a Game

- Before moving to the study of the most common types of games, we need to consider their foundamentals.
- The first important thing is to remember that players are rational. In other words, they:

1. are perfectly aware of the set of their potential actions;
2. form expectations on what is not known;
3. display clear preferences;
4. select the most appropriate action after a process of optimization.

Introduction to Games

Foundamentals of a Game

- Provided that there is no uncertainty, a model of rational behavior contains the following elements:
  1. A set $\mathscr{A}$ of actions;
  2. A set $\mathscr{C}$ of consequences linked to the above actions;
  3. A consequence function that links formally the consequence to the chosen action, $g : \mathscr{A} \to \mathscr{C}$;
  4. A *preference relation* $\succsim$ on the set of consequences $\mathscr{C}$.
- Regarding the consequence function, it is common to assume that it is a *utility function*, mapping the set of consequences into the real number set: $\mathscr{U} : \mathscr{C} \to \mathbb{R}$.
- Such an utility function defines the preference relation satisfying the condition that $x \succsim y$, *iff* $U(x) \succsim U(y)$.

- Given a subset of $\mathscr{A}$, say $\mathscr{B} \subseteq \mathscr{A}$, a rational player will choose an action a^*, which belongs to $\mathscr{B}$ and such that $g(a^*) \succsim g(a)$ $\forall a \in \mathscr{B}$
- Equivalently, we can say that a rational player finds the action a which maximize his utility function, i.e. $\max_{a \in \mathscr{B}} \mathscr{U}(g(a))$.
- Moreover individuals may take their decision in a situation of uncertainty about the game, the actions that other players can choose and so on.
- If this is the case we use a *Von Neumann-Morgenstern* utility: players maximize the expected utility that they drawn from a function, that assigns a specific number to each consequence.

- This happens if the consequence function is stochastic and known to each player. In this case she has a subjective probability distribution in mind, determining the consequence of each action.
- The individual considers a state space Ω, a probability measure over such Ω which links the action to the consequence, i.e. $\mathscr{A} \times \Omega \to \mathscr{C}$ and a utility function that maps the consequence into the real number space, i.e. $u : \mathscr{C} \to \mathbb{R}$.
- In this case each player maximize her (expected) utility function with respect to a given probability measure, i.e.:

$$\max_{a \in \mathscr{B} | \omega \in \Omega} u\left(g\left(a, \omega\right)\right). \tag{1}$$

- Although we try to keep mathematics at the minimum, some notation and terminology are necessary and useful for the rest of our discussion:
  - $\mathbb{R}$ is the set of real numbers and $\mathbb{R}_+$ is the subset of nonnegative real numbers. $\mathbb{R}^n$ is the set of vectors of n real numbers, while $\mathbb{R}_+^n$ is the set of vectors of n nonnegative numbers;
  - For any $x \in \mathbb{R}^n$ and any $y \in \mathbb{R}^n$, $x \geq y$ indicates that $x_i \geq y_i$ $\forall i = 1..n$;
  - A function $f : \mathbb{R} \to \mathbb{R}$ is increasing if $f(x) > f(y)$ whenever $x > y$ and nondecreasing if $f(x) \geq f(y)$ whenever $x > y$. A function $f : \mathbb{R} \to \mathbb{R}$ is concave if $f(\alpha x + (1-\alpha) x') \geq \alpha f(x) + (1-\alpha) f(x')$ $\forall x$ and $x' \in \mathbb{R}$ and any $\alpha \in [0,1]$.

- Further notation:
  - Given a function $f : X \to \mathbb{R}$, $\arg\max_{x \in X} f(x)$ denotes the set of maximizers for f;
  - $\mathscr{N}$ is the set of players. When we refer to a collection of values for any variable of a player (player profile), we use $(x_i)_{i \in \mathscr{N}}$ or (x_i) if there is no chance of confusion.
  - A binary relation $\succsim$ over $\mathscr{A}$ is complete if $a \succsim b$ or $b \succsim a$; it is reflessive if $a \succsim a$ and it is transitive if $a \succsim b$, $b \succsim c$ and $a \succsim c$ $\forall a$, b and $c \in \mathscr{A}$. If $a \succsim b$ but not that $b \succsim a$, then $a \succ b$. Instead if $a \succsim b$ and $b \succsim a$, we have that $a \sim b$

- Further notation:
  - A preference relation on a and b both belonging to $\mathscr{A}$ is continuous if $a \succsim b$ whenever it exists a sequence $\left(a^k\right)_k$ and $\left(b^k\right)_k$ that converge respectively to a and b and $a^k \succsim b^k$ for all k;
  - A preference relation is quasi-concave if for every $b \in \mathbb{R}^n$ the set $\{a \in \mathbb{R}^n : a \succsim b\}$ is convex. If the latter set is strictly convex, the preference relation is strictly quasi-concave.
  - For any profile $x = \left(x_j\right)_{j \in \mathscr{N}}$ and any $i \in \mathscr{N}$, x_{-i} is the profile referring to all other players except i. Alternatively $x_{-i} = \left(x_j\right)_{j \in \mathscr{N} \setminus \{i\}}$. Clearly, the profile (x_i, x_{-i}) is $(x_i)_{i \in \mathscr{N}}$. If X_i is a set for each $i \in \mathscr{N}$, then X_{-i} refers to $\times_{j \in \mathscr{N} \setminus \{i\}} X_j$; (More notation here)

Outline

Introduction

Introduction to Games

Strategic Form Games

- Perfect information
- Bayesian Games

Extensive Form Games

Repeated Games

Strategic Form Games

Introduction

- Strategic games (sometimes defined as *normal games*) characterize a situation in which players decide simultaneously their action once for all.
- Formally, we can represent a strategic game as follows:
  1. A finite set of players $\mathscr{N}$;
  2. For each $i \in N$ a non empty set A_i, containing all possible actions;
  3. For each player $i \in N$ a preference relation $\succsim_i$on $A = \times_{j \in N} A_j$.
- Point 3 indicates that the preference relation for individual i is defined over A, not over A_i, because of *strategic* thinking. Individual i does not take into consideration only her own actions, but also the choices made by other players.

- For instance, consider 3 firms, operating in an oligopoly. The set of possible actions is given by the set of potential prices that each firm can set.
- But is it really interesting for us simply to understand which price each firm will select? Possibly it is more interesting to know how that price tanslates into a firm's profit.
- Again, if some of the firms are unaware of some elements of the game, we need to take into consideration the probability distribution over some state of the world.
- Quite frequently, the consequence function is specified in terms of utility (**payoffs**). Therefore, a game can be represented in a compact way as follows: $\langle N, (A_i), (u_i) \rangle$.

Strategic Form Games

Formal Representation

- A strategic (normal) form game is represented as follows:

Strategic Form Games

Formal Representation

- A strategic (normal) form game is represented as follows:

		P2	
		L	M
P1	T	w_1, w_2	x_1, x_2
	B	y_1, y_2	z_1, z_2

Figure 1: A game in strategic form

- The easiest interpretation of a strategic game is to considere that the game will be played only once.
- Players choose their action simultaneously. Although they cannot be unware of what the other players will choose, they perfectly know the set of actions of the other players and which payoff each player will obtain when the game is played.
- Therefore, there is no any type of private information on which they can base their expectations.
- Another interpretation could be that a player can form her expectations on the basis of the way in which the game was played in the past.

Strategic Form Games

Interpretation

- A sequence of games can be modelled as a strategic game if is no link between each play. Players would care of the instantaneous payoff only.
- Players choose simultaneously, but they do not take a decision in exactly the same istant in time.
- **Simultaneity as independency**: Players take their action independently of what the other actually chooses.
- **Example**: two persons in front of a pc in two different rooms. They choose among some options (actions) and their choice will be recorderd by a central pc. They know the possible choices of each player and the payoffs. After both made their choices, the latter will be announced and payoffs assigned.

Strategic Form Games

Solution

- A solution is the tool that we will apply in order to determine the equilibrium of the game.
- We find out which action each player should take to be better off.
- Finding a solution means finding the steady state of the game, i.e. the situation from which players are not interested to deviate.
- It is not important which process brought players to the steady state. Instead, by choice the actions that determine the equilibrium, players set correctly their expectations on what others would have done and act rationally.
- The most powerful and famous solution concept is the Nash (1951) equilibrium.

Strategic Form Games

Iterated deletion of dominated strategies

- Another solution concept is the iterated deletion of dominated strategies.
- Formally, an action (strategy) a_i is (strongly) dominated if there exists another action $a_i^{'}$, such that:

$$\left(a_i^{'}, a_{-i}\right) \succsim_i (a_i, a_{-i}) \tag{2}$$

- However:

1. A dominated strategy does not necessarily exists;
2. After deletion of dominated strategies, the set of remaining actions characterize a Nash equilibrium.

Strategic Form Games

Iterated deletion of dominated strategies

- This approach consists of removing from the set of actions those that a player will never play since the correpsonding payoffs are always smaller than the ones obtained by playing alternative actions.
- Notice that after eliminating some strategies, others that were not dominanted at the outset, can be eliminated now.
- In a finite game, after a reasonable number of rounds, we can find the strategy that represents the equilibrium of the game.
- An example of this approach can be found, following this link:
  ▸ Deletion of dominared strategies

Strategic Form Games

Nash Equilibrium

- Let us consider the following definition:

Strategic Form Games

Nash Equilibrium

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Definition

A Nash equilibrium of a strategic game $\langle N, (A_i), \succsim_i \rangle$ is a profile $\alpha^* \in A$ of actions such that for every $i \in N$ we have:

Strategic Form Games

Nash Equilibrium

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Definition

A Nash equilibrium of a strategic game $\langle N, (A_i), \succsim_i \rangle$ is a profile $\alpha^* \in A$ of actions such that for every $i \in N$ we have:

$$\left(a^*_{-i}, a^*_i\right) \succsim_i \left(a^*_{-i}, a_i\right) \ \forall a_i \in A_i \tag{3}$$

Strategic Form Games

Nash Equilibrium

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Definition

A Nash equilibrium of a strategic game $\langle N, (A_i), \succsim_i \rangle$ is a profile $\alpha^* \in A$ of actions such that for every $i \in N$ we have:

$$\left(a^*_{-i}, a^*_i\right) \succsim_i \left(a^*_{-i}, a_i\right) \;\; \forall a_i \in A_i \tag{3}$$

- Interpretation: Given a set of possible actions, a player i do not have any action a_i that gives her an outcome preferable to that obtained when a^*_i is chosen, provided that the other players selected their equilibrium action a^*_{-i}.

- It can be useful to consider the following refinement. Let us assume that for each a_{-i}, $B\left(a_{-i}\right)$ is the set of best actions of player i conditional upon the choice made by other players, a_{-i}:

Strategic Form Games

Nash Equilibrium

- It can be useful to consider the following refinement. Let us assume that for each a_{-i}, $B\left(a_{-i}\right)$ is the set of best actions of player i conditional upon the choice made by other players, a_{-i}:

$$B_i\left(a_{-i}\right) = \left\{a_i \in A_i : \left(a_{-i}, a_i\right) \succsim_i \left(a_{-i}, a_i'\right) \ \forall a_i' \in A_i\right\} \qquad (4)$$

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- The set-valued function $B_i\left(a_{-i}\right)$ can be considered as the **best response function** of player i given that the other players will play their best action. Then, the Nash equilibrium is a profile a_i^* such that:

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- The set-valued function $B_i\left(a_{-i}\right)$ can be considered as the **best response function** of player i given that the other players will play their best action. Then, the Nash equilibrium is a profile a_i^* such that:

$$a_i^* \in B\left(a_{-i}\right) \ \forall i \in N \quad (5)$$

- The latter definition of the Nash equilibrium is useful because it offers us a simple way to find a Nash equilibrium in a game:

1. Calculate the best responses of each player;
2. Find the profile a^* of actions such that $a_i^* \in B_i\left(a_{-i}^*\right)\ \forall i \in N$;
3. If the functions B_i are singleton-valued, then finding the profile a^* consists of solving $|N|$ equations in $|N|$ unknowns.

- A very popular game is known as the **"Battle of Sexes"**:

Strategic Form Games

Nash Equilibrium - The Battle of Sexes

- A very popular game is known as the **"Battle of Sexes"**:

P1 \ P2	FB	FW
FB	$2,1$	$0,0$
FW	$0,0$	$1,2$

Figure 2: The Battle of Sexes

Strategic Form Games

- A very popular game is known as the **"Battle of Sexes"**:

P2

		FB	FW
P1	FB	$2,1$	$0,0$
	FW	$0,0$	$1,2$

Figure 2: The Battle of Sexes

- where FB stands for football and FW stands for fashion week.

Strategic Form Games

Nash Equilibrium - The Battle of Sexes

- The above game contains two Nash equilibria: FB, FB and FW, FW.
- The rationale of the game: to show how (and whether) people may coordinate themselves toward a common choice in the presence of conflicting interests.
- Before showing other examples, which is the procedure to find a Nash Equilibrium (NE)?
- Remember that for player i a NE is a profile a_i^* such that individual i cannot have an alternative action, a_i', that brings her a better outcome, when other players choose their equilibrium action.

Strategic Form Games

Nash Equilibrium - The Battle of Sexes

- Based on the above consideration:

1. Choose for instance player 1 and fix her strategy, i.e. suppose that she will play one of her available actions;
2. Find for player 2 the strategy that gives him the higher payoff;
3. Then suppose that player 1 plays the other(s) strategy(ies) and repeat step 2;
4. Every time you find the action that makes player 2 better off, underline the payoff connected with that action;
5. Repeat the above steps fixing player 2's actions and checking which is player 1's action that makes her better off;
6. When you have in the same box two underlined payoffs, that box indicates a Nash equilibrium.

- This is the solution of the game:

- This is the solution of the game:

		P2	
		FB	FW
P1	FB	$2,1$	$0,0$
	FW	$0,0$	$1,2$

Figure 3: The Battle of Sexes - Solution

Strategic Form Games

Nash Equilibrium - The Battle of Sexes

- This is the solution of the game:

P2

P1		FB	FW
	$\underline{FB}$	$2,1$	$0,0$
	FW	$0,0$	$1,2$

Figure 3: The Battle of Sexes - Solution

Strategic Form Games

Nash Equilibrium - The Battle of Sexes

- This is the solution of the game:

		P2	
		FB	FW
P1	$\underline{FB}$	$2, \underline{1}$	$0, 0$
	FW	$0, 0$	$1, 2$

Figure 3: The Battle of Sexes - Solution

Strategic Form Games

Nash Equilibrium - The Battle of Sexes

- This is the solution of the game:

		P2	
		FB	FW
P1	$\underline{FB}$	$2, \underline{1}$	$0, 0$
	$\underline{FW}$	$0, 0$	$1, 2$

Figure 3: The Battle of Sexes - Solution

Strategic Form Games

Nash Equilibrium - The Battle of Sexes

- This is the solution of the game:

		P2: FB	P2: FW
P1	$\underline{FB}$	$2, \underline{1}$	$0, 0$
P1	$\underline{FW}$	$0, 0$	$1, \underline{2}$

Figure 3: The Battle of Sexes - Solution

Strategic Form Games

Nash Equilibrium - The Battle of Sexes

- This is the solution of the game:

		P2	
		$\underline{FB}$	FW
P1	FB	$2, \underline{1}$	$0, 0$
	FW	$0, 0$	$1, \underline{2}$

Figure 3: The Battle of Sexes - Solution

- This is the solution of the game:

		P2	
		$\underline{FB}$	FW
P1	FB	$\underline{2}, \underline{1}$	$0, 0$
	FW	$0, 0$	$1, \underline{2}$

Figure 3: The Battle of Sexes - Solution

Strategic Form Games

Nash Equilibrium - The Battle of Sexes

- This is the solution of the game:

P2

		$\underline{FB}$	$\underline{FW}$
P1	FB	$\underline{2},\underline{1}$	$0,0$
	FW	$0,0$	$1,\underline{2}$

Figure 3: The Battle of Sexes - Solution

Strategic Form Games

Nash Equilibrium - The Battle of Sexes

- This is the solution of the game:

		P2	
		$\underline{FB}$	$\underline{\text{FW}}$
P1	FB	$\underline{2}, \underline{1}$	$0, 0$
	FW	$0, 0$	$\underline{1}, \underline{2}$

Figure 3: The Battle of Sexes - Solution

Strategic Form Games

Nash Equilibrium - The Battle of Sexes

- This is the solution of the game:

		P2	
		$\underline{FB}$	$\underline{FW}$
P1	$\underline{FB}$	$\underline{2}, \underline{1}$	$0, 0$
	$\underline{FW}$	$0, 0$	$\underline{1}, \underline{2}$

Figure 3: The Battle of Sexes - Solution

Strategic Form Games

Nash Equilibrium - The Battle of Sexes

- This is the solution of the game:

		P2	
		FB	FW
P1	*FB*	2, 1	
	FW		1, 2

Figure 3: The Battle of Sexes - Solution

Strategic Form Games

Nash Equilibrium - The Battle of Sexes

- In the example above:

1. Suppose that Player 1 chooses to watch a football match. In this case Player 2 will find more convenient to watch the match instead of going to the fashion week (this is because in the first case he gets a payoff equal to $1 > 0$);
2. Suppose that Player 1 chooses to go to the fashion week. In this case Player 2 will choose to go the fashion week as well, since this action will bring a payoff of $2 > 0$;
3. Repeat the same procedure by fixing the action for Player 2 and checking which choice will make Player 1 better off;
4. We will end up with the two Nash equilibria in the game.

Strategic Form Games

Theorem of Nash Equilibrium

- We will prove the existence of a Nash equilibrium in a formal but not too much technical way (for those interested, a more formal proof is in Fudenberg and Tirole, 1992).
- Notice that we cannot be guarantee that a Nash equilibrium always exists (although we will revise such a claim later).
- We may be interested in the formal proof, since the existence of an equilibrium indicates that our game is consistent with a steady state solution, i.e. with the hypothesis that once we find that equilibrium, players do not find convenient to move from it.
- The mathematical tool, which is used to show the existence of a Nash equilibrium, is the **Kakutani (1941) fixed point theorem**.

- To fully understand the proof, we need the following Lemma:

Strategic Form Games

Nash Equilibrium theorem

- To fully understand the proof, we need the following Lemma:

Lemma

Let X be a compact convex subset of $\mathbb{R}^n$ and let $f : X \to X$ be a set-valued function for which: (i) $\forall x \in X$ the set $f(x)$ is non-empty and convex; (ii) the graph f is closed. Then there exists a value of $x^* \in X$ such that $x^* \in f(x^*)$.

- To fully understand the proof, we need the following Lemma:

Lemma

Let X be a compact convex subset of $\mathbb{R}^n$ and let $f : X \to X$ be a set-valued function for which: (i) $\forall x \in X$ the set $f(x)$ is non-empty and convex; (ii) the graph f is closed. Then there exists a value of $x^* \in X$ such that $x^* \in f(x^*)$.

- The meaning of the graph f being closed is that if there exist two sequences $\{x_n\}$ and $\{y_n\}$ such that $y_n \in f(x_n)$ $\forall n$, then $x_n \to x$ and $y_n \to y$ and $y \in f(x)$.

- Before moving to the main result of the Nash (1950), let us define a preference relation $\succsim_i$ over A to be *quasi-concave* on A_i if for every $a^* \in A$ the set $\left\{a_i \in A_i : \left(a^*_{-i}, a_i\right) \succsim_i a^*\right\}$ is convex. Based on the above Lemma, we can state:

Strategic Form Games

Nash Equilibrium theorem

- Before moving to the main result of the Nash (1950), let us define a preference relation $\succsim_i$ over A to be *quasi-concave* on A_i if for every $a^* \in A$ the set $\left\{a_i \in A_i : \left(a^*_{-i}, a_i\right) \succsim_i a^*\right\}$ is convex. Based on the above Lemma, we can state:

Theorem

The strategic game $\langle N, (A_i), (\succsim_i)\rangle$ has a Nash equilibrium if for all $i \in N$: (i) the set A_i is a nonempty, convex subset of an Euclidean space; (ii) the preference relation $\succsim_i$ is continuos and quasi-concave on A_i.

- Formal Proof

Strategic Form Games

Further popular games

- Consider the following game:

- Consider the following game:

		P2	
		FB	TS
P1	FB	$2, 2$	$0, 0$
	TS	$0, 0$	$1, 1$

Figure 4: The Coordination Game

Strategic Form Games

Further popular games

- Consider the following game:

		P2	
		FB	TS
P1	FB	$2,2$	$0,0$
	TS	$0,0$	$1,1$

Figure 4: The Coordination Game

- where FB stands for football and TS stands for tennis

Strategic Form Games

Further popular games

- Another one:

Strategic Form Games

Further popular games

- Another one:

P1 \ P2	C	NC
C	$1,1$	$4,0$
NC	$0,4$	$3,3$

Figure 5: The Prisoners' Dilemma Game

Strategic Form Games

Further popular games

- Another one:

		P2	
		C	NC
P1	C	$1,1$	$4,0$
	NC	$0,4$	$3,3$

Figure 5: The Prisoners' Dilemma Game

- where C stands for confess and NC stands for not confess.

- Welfare implications?

Strategic Form Games

Further popular games

- Another one:

Strategic Form Games

Further popular games

- Another one:

		P2	
		Hawk	*Dove*
P1	*Howk*	0, 0	4, 1
	Dove	1, 4	3, 3

Figure 6: The Hawk-Dove Game

Strategic Form Games

Further popular games

- Another one:

- Another one:

P2

P1		*Head*	*Tail*
	Head	$1, -1$	$-1, \ 1$
	Tail	$-1, \ 1$	$1, -1$

Figure 7: The Matching Penny Game

Strategic Form Games

A further example - Cournot duopoly

- Consider the following problem:
  1. Two firms, 1 and 2, operate within a duopoly market;
  2. They need to choose their optimal quantity $q_i \in [0, \infty]$;
  3. The market clering price is:

Strategic Form Games

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where $Q = q_1 + q_2$.

Strategic Form Games

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Strategic Form Games

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  5. Find the Nash equilibrium.

Strategic Form Games

A further example - Bertrand duopoly

- Consider the following problem:
  1. Two firms, 1 and 2, operate in a duopoly market;
  2. They choose the price to which they sell their product, $p_i \in [0, \infty)$;
  3. The quantity demanded by consumers is:

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$$q_i\left(p_i, p_j\right) = a - p_i + bp_j \tag{8}$$

Strategic Form Games

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$$\pi_i = q_i\left(p_i, p_j\right) p_i - cq_i \tag{9}$$

Strategic Form Games

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  5. Find the Nash equilibrium.

Strategic Form Games

Mixed strategies

- In the above examples players choose to play one of their possible strategy (a ***pure streategy***) in a deterministic way.
- For instance, in the Battle of Sexes game each player will decide to play either FB or FW.
- However, players may decide to play a ***mixed strategy***, i.e. the choice of the action is regulated by a probabilistic rule.
- Moreover, in some cases (as the Matching Penny game) a pure strategy equilibrium may fail to exist.
- When we account for the possibility of a mixed strategy, we need to make a small change in the structure of the game we consider.

- Previously, we defined a game as a triple $\langle N, (A_i)\,(\succsim_i)\rangle$. Now we need to specify for each player's a preference relation over lotteries on A.
- Conventionally, we consider that the preference relation statisfies the assumption of von Neumann and Morgestern. Therefore we define the consequence function as $u_i : A \to \mathbb{R}$.
- We can model our game as a triple $\langle N, (A_i)\,(u_i)\rangle$.
- Starting from game in strategic form $G = \langle N, (A_i)\,(u_i)\rangle$, $\Delta\,(a_i)$ is the set of probability distribution over A_i.

Strategic Form Games

Mixed strategies

- Each element in $\Delta\left(a_i\right)$ represents a mixed strategy, which is an independent randomization.
- A profile $\left(\alpha_j\right)_{j\in N}$ of a mixed strategy determines a probability distribution over A.
- Now, if A_j is finite, given the independence of the randomization, the probability of an action $a=\left(\alpha_j\right)_{j\in N}$ is $\Pi_{j\in N}\alpha_j\left(a_j\right)$ and the evaluation of player i is given by $\sum_{a\in A}\left(\Pi_{j\in N}\alpha_j\left(a_j\right)\right)u_i\left(a\right)$.

- We can provide the following mixed extension og G, in which the set of actions for player i is given by the probability distributions over the potential actions:

Strategic Form Games

Mixed strategies

- We can provide the following mixed extension og G, in which the set of actions for player i is given by the probability distributions over the potential actions:

Definition

The mixed extention of the strategic game $\langle N, (A_i)\,(u_i)\rangle$ is the strategic game $\langle N, (\Delta(A_i))\,(U_i)\rangle$, where $\Delta(A_i)$ assigns a probability distribution over A_i and $U_i : \times_{j\in N}\Delta\left(A_j\right) \to \mathbb{R}$ assign to each mixed strategy an expected value under u_i of the lottery over A detemined by α.

Strategic Form Games

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- The above simply means that if a player plays a mixed strategy, she chooses to play a game where each pure strategy is player with a certain positive probability.

Strategic Form Games

Mixed strategies

- Based on the above definition, we can characterize the equilibrium notion for a Bayesian game as follows:

Strategic Form Games

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A mixed strategy Nash equilibrium of a strategic form game is the Nash equilibrium of its mixed extension.

Strategic Form Games

Mixed strategies

- Based on the above definition, we can characterize the equilibrium notion for a Bayesian game as follows:

Definition

A mixed strategy Nash equilibrium of a strategic form game is the Nash equilibrium of its mixed extension.

- Suppose that $\alpha^* \in \times_{j\in N}\Delta\left(A_j\right)$ is the mixed strategy Nash equilibrium of $G = \langle N, (A_i)\,(u_i)\rangle$ in which the mixed strategy α^* is degenerate in the sense that it assigns to an action $a_i^* \in A_i$ a probability of 1. Since $A_i \subseteq \Delta\left(A_i\right)$, the action profile a^* is a Nash equilibrium of the game G.

Strategic Form Games

Mixed strategies

- Moreover, suppose that a^* is a Nash equilibrium for G. Then assuming linearity in U_i in α_i there is no any probability distribution over actions in A_i such that player i can get an higher payoff than the one that it can be generated by playing a mixed strategy with probability 1.

- In other words, a Nash equilibrium in pure strategy is itself an equilibrium of the mixed extension, which is played with probability 1.

- The important result which emerges from the above discussion is the following:

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Theorem
Every finite strategy game has a mixed strategy Nash equilibrium.

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- A proof of the above theorem can be found, following this link: ▸ Formal Proof.

- In order to compute correctly a mixed strategy Nash equilibrium, the following Lemma is important:

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Lemma

Consider the finite strategic game $G = \langle N, (A_i)\,(u_i)\rangle$. *Then* $\alpha^* \in \times_{i\in N}\Delta\,(A_i)$ *is a mixed strategy Nash equilibrium of* G *if and only if* $\forall i \in N$, *every pure strategy in the support of* α^* *is a best response to* α^*_{-i}.

Strategic Form Games

Mixed strategies

- In order to compute correctly a mixed strategy Nash equilibrium, the following Lemma is important:

Lemma

Consider the finite strategic game $G = \langle N, (A_i)\,(u_i)\rangle$. Then $\alpha^* \in \times_{i \in N} \Delta(A_i)$ is a mixed strategy Nash equilibrium of G if and only if $\forall i \in N$, every pure strategy in the support of α^* is a best response to α^*_{-i}.

- Also in this case the proof of the lemma can be found here: ▸ Formal Proof.

Strategic Form Games

Mixed strategies - Matching Penny Game

- Consider again the Matching Penny game:

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		P2	
		Head	*Tail*
P1	*Head*	$1, -1$	$-1, \ 1$
	Tail	$-1, \ 1$	$1, -1$

Strategic Form Games

Mixed strategies - Matching Penny Game

- Consider again the Matching Penny game:

		P2	
		Head	*Tail*
P1	*Head*	$1, -1$	$-1, 1$
	Tail	$-1, 1$	$1, -1$

- We noted that in the above game, there is no a Nash equilibrium. This is true in pure strategies, but it might not be true if we consider mixed strategies.

Strategic Form Games

Mixed strategies - Matching Penny Game

- Now, let us suppose that player 1 believes that player 2 will play $Head$ with probability q and $Tail$ with probability $(1-q)$, i.e.:

Strategic Form Games

Mixed strategies - Matching Penny Game

- Now, let us suppose that player 1 believes that player 2 will play $Head$ with probability q and $Tail$ with probability $(1-q)$, i.e.:

P2

P1		$Head$ (q)	$Tail$ $(1-q)$
	$Head$ (r)	$1, -1$	$-1, \;\; 1$
	$Tail$ $(1-r)$	$-1, \;\; 1$	$1, -1$

Figure 8: Mixed Strategies in Matching Penny Game

- Given previous belief, we can easily calculate the expected payoff for player 1 if she decides to play either $Head$ or $Tail$:

Strategic Form Games

Mixed strategies - Matching Penny Game

- Given previous belief, we can easily calculate the expected payoff for player 1 if she decides to play either *Head* or *Tail*:

$$E\left(u_1\right)_{Tail} = q\left(-1\right) + \left(1-q\right)\left(1\right) = 1 - 2q \quad (10)$$

$$E\left(u_1\right)_{Head} = q\left(1\right) + \left(1-q\right)\left(-1\right) = 2q - 1 \quad (11)$$

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Strategic Form Games

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$$q\begin{cases} >\frac{1}{2} & Head > Tail \\ =\frac{1}{2} & Head \sim Tail \\ <\frac{1}{2} & Head < Tail \end{cases}$$

Strategic Form Games

Mixed strategies - Matching Penny Game

- Now, we need to consider the possible mixed strategy responses by player 1. Suppose that he will play $Head$ or $Tail$ with probabilities $(r, 1-r)$ and $Head$ with probability r. For each value of $q \in [0, 1]$ we can compute the values of r such that $(r, 1-r)$ is a best response to $(q, 1-q)$.
- Player's 1 expected payoff from playing $(r, 1-r)$ when player 2 plays $(q, 1-q)$ is:

Strategic Form Games

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- Player's 1 expected payoff from playing $(r, 1-r)$ when player 2 plays $(q, 1-q)$ is:

$$\begin{aligned} E(u_1) &= rq - r(1-q) - q(1-r) + (1-q)(1-r) \\ &= (1-2q) - r(2-4q) \end{aligned}$$

- The expected utility for player 1 is increasing in r provided that $2 - 4q < 0$, i.e. $q > \frac{1}{2}$ and decreasing otherwise.
- Therefore, player 1 will play $Head$ if $q > \frac{1}{2}$ and will play $Tail$ if $q < \frac{1}{2}$.
- By a similar argument, player 2 believes that player 1 will play $Head$ with probability r and $Tail$ with probability $(1 - r)$. Therefore, her expected payoff if she decides to play either $Head$ or $Tail$ is:

Strategic Form Games

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- By a similar argument, player 2 believes that player 1 will play *Head* with probability r and *Tail* with probability $(1-r)$. Therefore, her expected payoff if she decides to play either *Head* or *Tail* is:

$$E\left(u_2\right)_{Head} = r\left(-1\right)+\left(1-r\right)\left(1\right)=1-2r \qquad (12)$$
$$E\left(u_2\right)_{Tail} = r\left(1\right)+\left(1-r\right)\left(-1\right)=2r-1 \qquad (13)$$

Strategic Form Games

Mixed strategies - Matching Penny Game

- It is easy to note that:

Strategic Form Games

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$$r \begin{cases} < \frac{1}{2} & Head > Tail \\ = \frac{1}{2} & Head \sim Tail \\ > \frac{1}{2} & Head < Tail \end{cases}$$

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Mixed strategies - Matching Penny Game

- It is easy to note that:

$$r \begin{cases} < \frac{1}{2} & Head > Tail \\ = \frac{1}{2} & Head \sim Tail \\ > \frac{1}{2} & Head < Tail \end{cases}$$

- It follows that the mixed strategy Nash equilibrium is such that with probability $\frac{1}{2}$ both players will mix over their possible strategies.

Strategic Form Games

Mixed strategies - Matching Penny Game

- Below, a graphical representation of the mixed strategy equilibrium:

Strategic Form Games

Mixed strategies - Matching Penny Game

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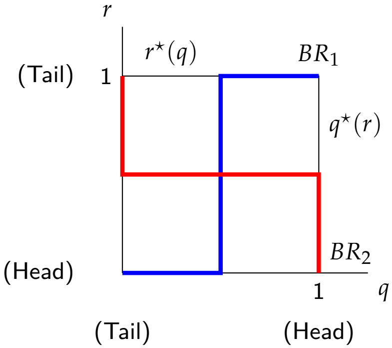

Figure 9: Matching penny game: the equilibrium

Strategic Form Games

Mixed strategies - Example

- Consider the Battle of Sexes game, we saw earlier:

Strategic Form Games

Mixed strategies - Example

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P2

		FB	FW
P1	FB	2, 1	0, 0
	FW	0, 0	1, 2

Strategic Form Games

Mixed strategies - Example

- Consider the Battle of Sexes game, we saw earlier:

P2

		FB	FW
P1	FB	$2,1$	$0,0$
	FW	$0,0$	$1,2$

- Is there any equlibrium in mixed strategies, a part from the one in which FB, FB and FW, FW are played both with probability 1?

- Let us consider now another type of game where players do not share the same type of information.
- In this type of games, some players are not aware of the characteristics of the other players.
- As for the previous games, we consider a set of N players and a set of actions A_i.
- We introduce players' uncertainty about each other by introducing a set Ω of possible states of nature. Each state describes different characteristics for each player.

Strategic Form Games

Bayesian Games

- Each player has some **prior belief** about the state of the nature given by p_i on Ω.
- In any given play of the game some state of nature $\omega \in \Omega$ is realized.
- We can model players' information about the state of the nature through a profile τ_i of a signal function $\tau_i(\omega)$. This is a signal that a player i observes before choosing her action, when the state of nature is ω.
- T_i is the set of all possible values of τ_i and T_i can be understood as being the set of types for player i.

- Player i assigns a positive prior probability to each type in T_i and this probability if given by $p_i\left(\tau_i^{-1}(t_i)\right) > 0\ \forall t_i \in T_i$.
- If a player receives a signal $t_i \in T_i$, then she believes to be in the state $\tau_i^{-1}(t_i)$.
- Her posterior belief about the state of nature assigns to each state $\omega \in \Omega$ the probability $p_i(\omega) / p_i\left(\tau_i^{-1}(t_i)\right)$ if $\omega \in \tau_i^{-1}(t_i)$ and 0 otherwise.

Strategic Form Games

Bayesian Games

- While a player can know each possible action in each possible state ω, she does not know (a, ω) (she does not know in which state of nature she is).
- We can introduce a preference relation ($\succsim_i$) over the lotteries $A \times \Omega$. We can characterize a Bayesian game as:

1. A finite set of players N;
2. A finite set of possible states Ω;
3. A set of possible actions A_i;
4. A finite set of T_i, i.e. a finite set of signals that can be observed and a function $\tau_i : \Omega \to T_i$
5. A preference relation $\succsim_i$ on the set of probability measures over $A \times \Omega$, where $A = \times_{j \in N} A_j$.

Strategic Form Games

Bayesian Games

- **Important point:** the Bayesian game can be reduced to encompass simply the types of players.
- Each player i in the game knows her type. Therefore, she does not need to think of a possible plan in the even of being of some other type.
- However, this does not imply that an equilibrium should be defined in each state of nature in isolation. In other words, player i should take into consideration:

  1. What other players may decide to do in different states of nature;
  2. The formation of belifs depends also on the action that the player will choose in other states, since other players are unware about the state of nature.

Strategic Form Games

Bayesian Games

- In a Bayesian game each player chooses her best action given the signal she receives and her belief about the state of nature:

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Definition

A Nash equilibrium of a Bayesian game $\langle N, \Omega, (A_i), (T_i), (\tau_i), (p_i), (\succsim_i) \rangle$ is the Nash equilibrium of a strategic form game defined by:

1. the set of players in each state (i, t_i) $\forall i \in N$ and $t_i \in T_i$;
2. the set of actions of each player (i, t_i);
3. the preference ordering $\succsim^*_{(i,t_i)}$ defined by $a^* \succsim^*_{(i,t_i)} b^*$ iff $L_i(a^*, t_i) \succsim_i L_i(b^*, t_i)$ where $L_i(.)$ is the lottery over $A \times \Omega$ that assigns a probability $p_i(\omega) / p_i\left(\tau_i^{-1}(t_i)\right)$ to $\left(\left(a^*(j, \tau_j(\omega))\right)_{j \in N}, \omega\right)$ if $\omega \in \tau_i^{-1}(t_i)$ and 0 otherwise.

- **Interpretation:** In a Bayesian game each player chooses her best action given the signal that her receives and her belief about the state of nature, which is deducted by the signal she obtains.
- In a reduced version of the Bayesian game, the player may simply be unaware of the type of his opponent.
- Although such a simplification does not imply a loss of generality, it is preferable since it removes some (illusory) dynamics in the representation of the game.

- Consider the following game:
  1. Firms 1 and 2 form the set of players;
  2. Each firm chooses the amount of $q\,[0,\infty]$ in order to maximize its profit;
  3. Firm 1's cost function is $C_1 = cq_1$. Firm 2 can face either an high cost ($C_2^H = c_H q_2$) or a small one ($C_2^L = c_L q_2$).
  4. Firm 1 does not know Player 2's type, but with probability θ the cost will be C_2^H and with probability $(1-\theta)$ it is C_2^L.
  5. The above elements are common knowledge among players. Therefore, Firm 2 has a superior information and it knows that Firm 1 knows this.
  6. Firm 2 chooses a quantity, which is consistent with its cost type. But it knows that Firm 1 may anticipate this.

- Given the above characteristics of the game:

  1. Could you write down the profit functions for firm 1 and firm 2?
  2. Could you characterize the quantity that a firm select in equilibrium?

Outline

Introduction

Introduction to Games

Strategic Form Games

Extensive Form Games

- Introduction
- Characteristics
- Example
- Strategies
- Solution Concepts
- Subgame Perfect Equilibrium - SPE

Repeated Games

Extensive Form Games

Introduction

- Let us consider another class of games. Suppose to have a structure of the game such that it is sequential.
- In the development of this type of game, we assume perfect information in the sense that individuals know what happened in the past.
- The above assumption (perfect information) can be relaxed, but we will not study this type of games.

Extensive Form Games

Definition

An extensive form game with perfect information has the following structure:

1. A set of players N;
2. A set of H sequences satisfying the following properties:
   - 2.1 The empty sequence $\varnothing$ is a part of the sequence H;
   - 2.2 If $\left(a^k\right)_{k=1,..,K} \in H$ and $L < K$, then $\left(a^k\right)_{k=1,..,L} \in H$;
   - 2.3 If an infinite sequence $\left(a^k\right)_{k=1}^{\infty}$ satisfies $\left(a^k\right)_{k=1,..,L} \in H$, then $\left(a^k\right)_{k=1}^{\infty} \in H$;
3. A function P which gives to each nonterminal history $(H \backslash Z)$ a member in N;
4. For each

Extensive Form Games

Characteristics

- In the above definition H reflects the **history** of the game. Each element in the history is an action taken by a player.
- Consider that a history $\left(a^k\right)_{k=1,..,K} \in H$ is **terminal** if there is no any further action, a^{K+1} such that the sequence $\left(a^k\right)_{k=1,..,K+1} \in H$. In other words, this indicates that the game reached an end (or that it is finite). The set of terminal histories is indicated as Z.
- P is the player function, i.e. it is the function that assigns to each nonterminal history a player, that has to take an action.

Extensive Form Games

Characteristics

- If the set of all possible history is finite, also the extensive game is finite. If the longest history is finite, also the game is finite.
- Suppose an history of length k. We can denote by (h, a) the history at time $k+1$, which will be followed by a.
- HINT: It is convenient to interpret history as the sequence of moves that occur in an extensive game, which encompass the existence of some "dynamics".

Extensive Form Games

Characteristics

- The interpretation of the extensive form game is the following. After each nonterminal history h a player $P(h)$ chooses his action from the set $A(h) = \{a : (h, a) \in H\}$.

- The empty history $\varnothing$ is the initial point of the history (i.e. the point in time at which the game starts).

- In the next slide, we can consider an example of a game in extensive form.

Extensive Form Games

Example

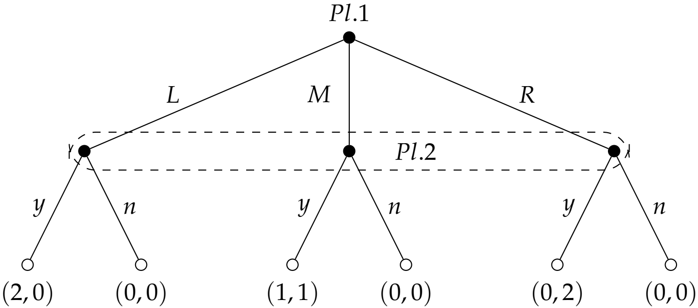

Figure 10: A game in extensive form

Extensive Form Games

Example

- The above figure depicts an easy and convenient way to characterize an extensive game as a tree.
- Each branch indicates an action that a player can take.
- A game starts with an empty sequence $\varnothing$ at which player 1 moves, i.e. $P(\varnothing) = 1$. The three branches departing from that node, $A(\varnothing)$, indicates the set of actions that player 1 can take at that stage of the game.
- Numbers at the end of the game (after player 2 moves) indicate the payoffs that both obtain.
- The game consists of player 1 setting an offer on how to divide a cake of size 2 and player 2 accepting or rejecting that offer.

- Let us consider the following definition:

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Definition

A strategy for $\forall i \in N$ in an extensive form game with perfect information $\langle N, H, P, (\succsim_i) \rangle$ is a function that gives an action in $A(h)$ to each nonterminal history $h \in H \backslash Z$ for which $P(h) = i$.

- Let us consider the following definition:

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A strategy for $\forall i \in N$ in an extensive form game with perfect information $\langle N, H, P, (\succsim_i) \rangle$ is a function that gives an action in $A(h)$ to each nonterminal history $h \in H \backslash Z$ for which $P(h) = i$.

- To understand the content of the above definition, we can consider the following:
  1. Player 1 moves first and proposes a given partition of the cake - $(2,0)$, $(1,1)$ or $(0,2)$;
  2. Player 2 chooses one of his actions (y, n).
- Player 2 strategies are a triple $a_2 b_2 c_2$ corresponding to an action conditional upon player 1's choice.

- The interpretation of the above triple is simple. Player 2 needs to be prepared for any contingency. Therefore:
  1. If player 1 plays $(2,0)$, he will play a_2,
  2. If player 1 plays $(1,1)$, he will play b_2,
  3. If player 1 plays $(0,2)$, he will play c_2.
- The implication of the above interpretation can be better understood if we consider the following game.

Extensive Form Games

Another example

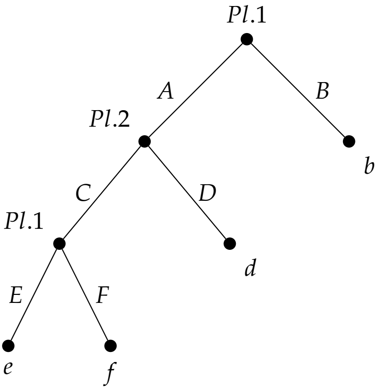

Figure 11: Another game in extensive form

Extensive Form Games

Strategies

- The interpretation of the above triple is simple. Player 2 needs to be prepared for any contingency. Therefore:

  1. In the above game, player 2 should specify a strategy, even for histories that cannot be reached.
  2. More specifically, player 1 strategies are $\{AE, AF, BE, BF\}$, while for player 2 are $\{C, D\}$.
  3. Player 1 specifies a strategy for the history (A, C), even though it cannot be reached if she chooses B at his first move.
  4. From this, we can observe the difference between a plan of actions as we saw before and a set of strategies.

- Based on what we noticed above, the following definition applied:

- Based on what we noticed above, the following definition applied:

Definition

Given a strategy profile $s = (s_i)_{i \in N}$ in $\langle N, H, P, (\succsim_i) \rangle$, the outcome $O(s)$ of s is the terminal history that follows the development of the game according to s_i. In other words, $O(s)$ is the history $\left(a^1, .., a^K\right) \in Z$, such that for $0 \leq k < K$, we have $s_{P\left(a^1,..,a^k\right)}\left(a^1, .., a^k\right) = a^{k+1}$.

Extensive Form Games

Solution Concepts

- We may consider two solution concepts to find an equilibrium. The first is the Nash equilibrium:

- We may consider two solution concepts to find an equilibrium. The first is the Nash equilibrium:

Definition

A Nash equilibrium of an extensive form game with perfect information $\langle N, H, P, (\succsim_i)\rangle$ is a strategy profile s^* such that $O\left(s^*_{-i}, s^*_i\right) \succsim_i O\left(s^*_{-i}, s_i\right)\ \forall i \in N$.

- An alternative way to think about the above equilibrium is to consider the equilibrium of the transformation of the extensive to the strategic form game. More specifically:

- An alternative way to think about the above equilibrium is to consider the equilibrium of the transformation of the extensive to the strategic form game. More specifically:

Definition

Let Γ be an extensive form game $\langle N, H, P, (\succsim_i)\rangle$. The game $\left\langle N, (S_i), \left(\succsim_i^{'}\right)\right\rangle$ is the strategic form of game Γ where S_i is the set of strategies of player i in Γ and $\succsim_i^{'}$ is defined by $s \succsim_i^{'} s^{'}$ iff $O(s) \succsim_i^{'} O\left(s^{'}\right)$.

- The last definition is quite interesting, since it points out the possibility that an extensive game can be transformed into a strategic game.

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P1 \ P2	C	D
AE	e	d
AF	f	d
BE	b	b
BF	b	b

Figure 12: From extensive to strategic form

Extensive Form Games

Solution Concepts

- The definition of equilibrium given above can lead to some questionable results. Let us consider the following game:

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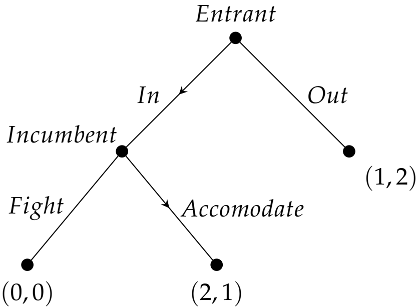

Figure 13: Predation Game

- The above game contains two Nash equilibria: $\{(In, Accom)\ (Out, Fight)\}$. Are both equilibria **plausible**?
- Consider the following reasoning:
  1. Player 1 clearly prefers the equilibrium $(In, Accom)$ to the one $(Out, Fight)$, since in this case he gets a payoff of $2 > 1$;
  2. Instead, player 2 prefers $(Out, Fight)$ to $(In, Accom)$;
  3. The sustainability of the first equilibrium is based on the threat that player 2 can play $Fight$ after player 1 choosing In.
  4. But player 1 knows that such a threat is not **credible**, since if player 2 plays $Fight$ instead of $Accom$, the latter would receive 0 instead of 1.

Extensive Form Games

Subgame Perfect Equilibrium - SPE

- We can discard one of the equilibria, since it is not logically consistent with the objective of each player to maximize her own utility.
- In order to remove the possibility of such equilibria, we can use a different solution concept. First, let us consider the following definition:

Extensive Form Games

Subgame Perfect Equilibrium - SPE

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- In order to remove the possibility of such equilibria, we can use a different solution concept. First, let us consider the following definition:

Definition

A subgame of the extensive game with perfect information $\Gamma = \langle N, H, P, (\succsim_i) \rangle$ that follows the history h is itself an extensive form game $\Gamma(h) = \langle N, H|_h, P|_h, (\succsim_i |_h) \rangle$. $H|_h$ is the set of sequences h' of actions for which $\left(h, h'\right) \in H$, while $P|_h$ is $P|_h\left(h'\right) = P\left(h, h'\right)$ $\forall h' \in H|_h$ and $(\succsim_i |_h)$ is defined as $h' \succsim_i |_h h''$ iff $\left(h, h'\right) \succsim_i \left(h, h''\right)$.

- The above definition can be interpreted as follows. A subgame is a part of the entire game, which is reached given a particular history of the game itself.
- Players will play such a subgame and choose their best strategy profile. The following definition follows:

Extensive Form Games

Subgame Perfect Equilibrium - SPE

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Extensive Form Games

Subgame Perfect Equilibrium - SPE

- The above definition can be interpreted as follows. A subgame is a part of the entire game, which is reached given a particular history of the game itself.

- Players will play such a subgame and choose their best strategy profile. The following definition follows:

Definition

A subgame perfect equilibrium (SPE) of an extensive form game with perfect information $\Gamma = \langle N, H, P, (\succsim_i) \rangle$ is a strategy profile s^* such that for every player $i \in N$ and every nonterminal history $h \in H \backslash Z$ for which $P(h) = i$, we have that $O_h\left(s^*_{-i}|_h, s^*_i|_h\right) \succsim_i |_h O_h\left(s^*_{-i}|_h, s_i|_h\right)$.

- The advantage to use such a solution concept is that it allows us to discard equilibria that are not credible as in the case we saw above.
- To verify that a strategy profile s^* is a SPE we need to check that for each player and subgame, there is no a strategy, which guarantees a better payoff.
- It can be shown that in a finite game we can simply pay attention to the alternative strategies (different from s^*) that follow the first history.
- In other words, a strategy profile is a SPE iff for each subgame the player that moves first cannot improve her utility simply changing only her initial action.

- The last outlined result brings useful insight. Let us keep in mind that a strategy profile is a SPE iff for each subgame the player that moves first cannot improve her utility simply changing only her initial action.
- Then, relying on the previous claim, it is possible to show that any finite extensive form game has a SPE:

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- The last outlined result brings useful insight. Let us keep in mind that a strategy profile is a SPE iff for each subgame the player that moves first cannot improve her utility simply changing only her initial action.
- Then, relying on the previous claim, it is possible to show that any finite extensive form game has a SPE:

Proposition

Every finite extensive game with perfect information has a subgame perfect equilibrium.

Extensive Form Games

Subgame Perfect Equilibrium - SPE

- Rather than producing a formal proof , we can give its intuition (for a formal proof, see Osborne and Rubinstein, 1994, p. 99).
- If a SPE is a strategy profile such that the player that moves first cannot obtain more by changing her initial action, then we can say that in all histories before the terminal one the player who is supposed to move chooses her best strategy.
- Assuming that all players chose their best strategy, we consider which is the optimal strategy at the terminal history. Then, we can work our way back up to the start of the game.
- Such a result is known as Kuhn's theorem and the procedure that we apply is called **backward induction**.

Extensive Form Games

Stackelberg Game

- Consider the following scenario:

1. There are two firms competing in a duopoly market. Firms choose quantity as in a Cournot game.
2. Their possible strategy is a quantity $q_i \in [0, \infty)$.
3. Differently from a standard Cournot model, in the Stackelberg game firm 1 chooses its quantity and, after observing q_1, firm 2 will decide its strategy.
4. Also in this case, we can consider the **strategic effect**. Firm 2 observes the quantity chosen by its opponent, but firm 1 may anticipate firm 2 choice.

Extensive Form Games

Stackelberg Game

- The structure of the game is the following:
  1. $N = \{\text{Firm 1, Firm 2}\}$;
  2. $s_i = q_i \in [0, \infty)$;
  3. $\pi_i = q_i [P(Q) - c]$ where $Q = q_1 + q_2$;
- The timing of the game is the following:
  - Firm 1 chooses q_1;
  - Firm 2 observes q_1 and chooses q_2.
- Which is the SPE for this game? Can you derive any welfare implication for each firm and the market?
- After your analysis, confirm the statement: "In a multi-person decision problem, holding more information always makes an individual better off". Motivate your answer.

Outline

Introduction

Introduction to Games

Strategic Form Games

Extensive Form Games

Repeated Games
- Introduction
- Characteristics of an Infinite Game
- Trigger Strategy Equilibrium

Repeated Games

Introduction

- Let us now focus on another class of games, which are the repeated ones.Let us consider the following game:

- Let us now focus on another class of games, which are the repeated ones.Let us consider the following game:

		P2: *Steal*	P2: *Not Steal*
P1	*Steal*	1,1	5,0
	Not Steal	0,5	4,4

Figure 14: Steal/Not Steal Game

- The game in the above matrix aims at analysing the interactions among players, when a game is repeated infinitely.

Repeated Games

Introduction

- More specifically, the game aims at checking whether there exists the possibility that players "learn" to play a mutually desired outcome (which is in this case $\{Not\ Steal, Not\ Steal\}$).
- A convenient way to think about the game: Players live in a state of nature.
- The absence of rules naturally drives a player to steal her opponents' wealth.
- Since a punishment is not in place, players may not think of the possibility to leave in peace.
- The theory of repeated game indicates us the conditions under which the mutually beneficial equilibrium {Not steal, Not steal} can be sustained as a long run equilibrium.

- The possibility that this occurs, however, requires the existence of a punishment to be applied when an individual deviates from the appropriate behavior.
- In addition, the punishers should have an incentive to carry out such a punishment.
- The solution of such games consists in the application of the so called *Folk Theorems*, which show a double property:
  1. Outcomes that are socially desirable can emerge in the long run, although it is unlikely that this occurs in the short term;
  2. The set of equilibrium outcomes could be very large and, therefore, the notion of equilibrium lacks predictive power.

Repeated Games

Introduction

- The potential of such games can be acknowledged if we think are a way to show the existence of some social norms, that are stable.
- Repeated games can be either finite or infinite. Such a characteristic is not determined simply by the (relative) time length of the game, but by the number of times the game is played (even if the time length is equal to 1 day, the number of times a game is played could be very large and approximate the concept of infiniteness).
- Choosing the structure of the game is not a trivial task, since equilibria may be different when the game takes the first or the second structure.

- Let us consider an infinite game: players infinitely repeat a game like the one in the previous Figure.
- In other words, players repeatedly engage in a strategic game G that is played simultaneously. G is defined as a constituent game.
- Whenever the constituent game is played, it is a *stage* of the infinite game.
- When taking an action, each players knows what has been done in the past.

- The game is modelled as follows:

- The game is modelled as follows:

Definition

Let $G = \langle N, (A_i), (\succsim_i) \rangle$ be a strategic game with $A = x_{i \in N} A_i$. An infinite repeated game of G is an extensive form game with perfect information and simultaneous moves $\langle N, H, P, (\succsim_i^*) \rangle$ where: (i) $H = \{\varnothing\} \cup \left(\cup_{t=1}^{\infty} A^t \right) \cup A^\infty$ (where $\varnothing$ is the starting point of the game and A^∞ is the infinite sequence of actions $\left(a^t\right)_{t=1}^{\infty}$ in G); (ii) $P(h) = N$ for each nonterminal history $h \in H$; (iii) $\left(\succsim_i^*\right)$ is the preference relation on A^∞ that extents the preference relation $(\succsim)$ in the sense that it satisfies the condition of weak separability (if $\left(a^t\right) \in A^\infty$, $a \in A$, $a' \in A$ and $a \succsim_i a'$ then $\left(a^1, .., a^{t-1}, a, a^{t+1}, ...\right) \succsim_i^* \left(a^1, .., a^{t-1}, a', a^{t+1}, ...\right)$.

- A strategy profile for each player assigns an action in A_i to every finite sequence of outcomes in G.
- In addition to the above characteristics, we will assume that the preference relation is based on a payoff function u_i that represents preference relations for player i ($\succsim_i$) in G.
- Therefore, $\left(a^t\right) \succsim \left(b^t\right)$ simply depends on the relation between the corresponding sequences $u\left(a^t\right) \succsim u\left(b^t\right)$ in G.
- A further clarification refers to the form of the preference relation. We consider the discounting form.

Repeated Games

Characteristics of an Infinite Game

- The discounting form consists of assuming that there exists a value $\delta \in (0, 1)$ such that the sequence of real number $\left(v^t\right)$ is at least as good as $\left(w^t\right)$ iff $\sum_{t=1}^{\infty} \delta^{t-1} \left(v^t - w^t\right) \geq 0$.
- δ is the discount factor, which indicates the value that each player assigns to the future utility. A further interpretation of δ is that it measures the degree of patient of players: the closer δ to 1, the more patient the player is.
- When the preference relation assumes the above form, we refer to the preference relation $\left((1-\delta) \sum_{t=1}^{\infty} \delta^{t-1} v^t\right)_{i \in N}$ as the payoff profile in the repeated game which is associate to the sequence $\left(v^t\right)_{t=1}^{\infty}$ of payoffs in G.

Repeated Games

Characteristics of an Infinite Game

- We also need to say something more about the type of punishment that players may implement.
- They could take different forms. For instance, the punishment can last for some stages of the game or it could be infinite.
- The latter is also known as a *trigger strategy*. It consists of punishing any deviation from the mutually desirable outcome by implementing the non-cooperative actions infinitely.
- We will focus on such a strategy in our search for an equilibrium.

Repeated Games
Trigger Strategy Equilibrium

- We can describe the trigger strategy equilibrium as follows:
  1. Players agree to play the mutually beneficial strategy in all stages of the repeated game (in the example in the previous Figure this strategy is"*Not Steal*");
  2. They discount the future utilities by a discount factor $\delta \in (0, 1)$;
  3. If one player deviates from the agreed strategy (and it chooses the strategy "*Steal*"), the opponent will play the strategy "*Steal*" forever;
  4. In order to ensure that the mutually beneficial strategy is implemented forever, we need to find the value of the discount factor, $\underline{\delta}$, such that players do not have an incentive to deviate.

Repeated Games

Trigger Strategy Equilibrium

- Let $\{NS, S\}$ denote the set of strategies "*Not Steal*" and "*Steal*".
- Let us define with z the payoff that player i obtains when both players play "*Not Steal*", with m the payoff if both play "*Steal*" and with n the payoff if player i plays "*Steal*" and player j plays "*Not Steal*".
- If both players play the strategy NS forever, player i will obtain:

Repeated Games

Trigger Strategy Equilibrium

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- If both players play the strategy NS forever, player i will obtain:

$$u_i^{\{NS,NS\}} = \sum_{t=1}^{\infty} \delta^{t-1} z$$

- Instead, suppose that a player deviates from the mutually beneficial strategy.
- In this case, at the first stage player i plays NS and player j plays S. From now on both players will play S. Therefore, player i payoff would be:

Repeated Games

Trigger Strategy Equilibrium

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- In this case, at the first stage player i plays NS and player j plays S. From now on both players will play S. Therefore, player i payoff would be:

$$u_i^{\{S,NS\}} = m + \sum_{t=1}^{\infty} \delta^t n$$

- Player i does not have any incentive to deviate iff:

- Player i does not have any incentive to deviate iff:

$$\sum_{t=1}^{\infty} \delta^{t-1} z \geq m + \sum_{t=1}^{\infty} \delta^{t} n$$

- Given the above information, which is the value of δ such that players do not have any incentive to deviate in the game displayed in the game displayed above?

Additional Mathematical Notation

- Further notation:
  - For any $Y \subseteq X$, $f(Y)$ is the set $\{f(x) : x \in Y\}$;
  - Consider a set X. $|X|$ indicates the number of players in X. Then a partition of X is a collection of disjoint subsets of X, whose union gives X itself. Let $X \subseteq \mathbb{R}^n$ and $x \in X$. Then x is Pareto efficient if there is no any other $y \in X$ such that $y_i > x_i \ \forall i \in N$. x is strictly Pareto efficient if there is no any $y \in X$ such that $y_i \geq x_i \ \forall i \in N$;
  - A probability measure μ over a finite set X is additive in the sense that $\mu(A \cup B) = \mu(A) + \mu(B)$, provided that A and B are disjoint set. Back

Strategic Form Games

Iterated deletion of dominated strategies - Example

- Consider the following example:

Strategic Form Games

Iterated deletion of dominated strategies - Example

- Consider the following example:

		P2		
		L	C	R
	U	$0,2$	$3,1$	$2,3$
P1	M	$1,4$	$3,2$	$4,1$
	D	$2,1$	$4,4$	$3,2$

Figure A: Iterated deletion of dominated strategies

- Consider the following example:

P2

P1		L	C	R
	M	1, 4	3, 2	4, 1
	D	2, 1	4, 4	3, 2

Figure A: Iterated deletion of dominated strategies

- Consider the following example:

		P2		
		L	C	
P1	M	$1,4$	$3,2$	
	D	$2,1$	$4,4$	

Figure A: Iterated deletion of dominated strategies

Strategic Form Games

Iterated deletion of dominated strategies - Example

- Consider the following example:

			P2	
		L	C	
P1				
	D	$2, 1$	$4, 4$	

Figure A: Iterated deletion of dominated strategies

Strategic Form Games

Iterated deletion of dominated strategies - Example

- Consider the following example:

P2

			C	
P1				
	D		$4, 4$	

Figure A: Iterated deletion of dominated strategies

Strategic Form Games

Iterated deletion of dominated strategies - Example

- Consider the following example:

P2

			C	
P1				
	D		$4,4$	

Figure A: Iterated deletion of dominated strategies

Back

Strategic Form Games

Formal proof of Nash Equilibrium theorem

- Here is the proof:

Strategic Form Games

Formal proof of Nash Equilibrium theorem

- Here is the proof:

Proof.
First we define $B : A \to A$ as $B(a) = \times_{i \in N} B_i(a_{-i})$, i.e. as the set of best responses of i. $B_i(a_{-i})\ \forall i \in N$ is nonempty since $\succsim_i$ is continuous and A_i is compact. It is also convex since $\succsim_i$ is quasi-concave on A_i. Also by previous Lemma, B has a closed graph since $\succsim_i$ is continuous. It follows that by Kakutani fixed point theorem B has at least a fixed point. The fixed point is a Nash equilibrium of the game. □

Back

- Here is the proof:

Strategic Form Games

Formal proof of Nash Equilibrium theorem in mixed strategy

- Here is the proof:

Proof.

Consider the game $G = \langle N, (A_i)\,(u_i)\rangle$ and for each player i define m_i as being the number of actions in A_i. Then $\Delta\,(A_i)$ identifies the set of vectors $(p_i ... p_{m_i})$ such that $p_k \geq 0$ and $\sum_{k=1}^{m_i} p_k = 1$, where p_k is the probability that player i will play the kth pure strategy. This set is compact, convex and nonempty. Since expected payoffs are linear in the probabilities, each payoff function in the mixed extension of the game G is continuous and quasi-concave. Therefore, the mixed extention of G satisfies all the assumption required for having a Nash equilibrium. □

▸ Back

Strategic Form Games

Formal proof of Lemma

- Here is the proof:

Strategic Form Games

Formal proof of Lemma

- Here is the proof:

Proof.
First, suppose that there exists an action a_i in the support of α_i^* such that it is not a best response to α_{-i}^*. Then by linearity of U_i in α_i, player i can increase her payoff by playing another strategy a_i with a certain positive probability (in other words, player i can choose to assign a larger probability to another strategy). Therefore α^* cannot be a best response to α_{-i}^*. However this cannot be the case by definition of a_i^*. □

Strategic Form Games

Formal proof of Lemma

- Proof continued:

Strategic Form Games

Formal proof of Lemma

- Proof continued:

Proof.
Second suppose that there is a mixed strategy α_i' such that player i can obtain an higher payoff than the one he receives by playing α_i^* in response to α_{-i}^*. Again, by linearity of U_i there must exists one action in the support of α_i' that gives an higher payoff than some actions in the support of α_i^*. This implies that the actions in the support of α_i^* cannot be best responses to α_{-i}^*. Again this is in contrast with the definition of α_i^*. □

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